Multiscale theory of turbulence in wavelet representation

M.V.Altaisky
Joint Institute for Nuclear Research, Dubna, 141980, Russia;
and Space Research Institute, Profsoyuznaya 84/32,
Moscow, 117997, Russia; e-mail: altaisky@mx.iki.rssi.ru

June 7, 2004

Abstract

We present a multiscale description of hydrodynamic turbulence in incompressible fluid based on a continuous wavelet transform (CWT) and a stochastic hydrodynamics formalism. Defining the stirring random force by the correlation function of its wavelet components, we achieve the cancellation of loop divergences in the stochastic perturbation expansion. An extra contribution to the energy transfer from large to smaller scales is considered. It is shown that the Kolmogorov hypotheses are naturally reformulated in multiscale formalism. The multiscale perturbation theory and statistical closures based on the wavelet decomposition are constructed.

PACS: 47.27.Gs, 11.10.Gh

1 Introduction

The statistical description of a fully developed hydrodynamic turbulence is based on the Kolmogorov hypotheses [1] on the self-similarity of the velocity fluctuations of different scales. However, the Kolmogorov-Obukhov analysis [2, 3] does not provide a rigorous mathematical definition of the "fluctuation of scale l". In the literature this is tacitly understood as Fourier components with wavenumbers approximately equal to the inverse scale $k \approx \frac{2\pi}{l}$ and the analysis is performed in wavenumber space. This definition meets global characteristics of the fully developed isotropic turbulence, but, being based on the Fourier transform, is essentially nonlocal and therefore hardly applicable to such important properties of the fully developed turbulence as the coherent structure formation. To catch the local properties of

the turbulent velocity field, the decomposition into localized wave packets and wavelets have been performed by many authors [4, 5, 6, 7, 8, 9].

Most of the wavelet applications to turbulence are restricted either to an analysis of the measured turbulent fields with the "wavelet microscope", capable of simultaneous analysis of the same velocity field at different resolution [7, 8, 9], or to a numerical solution of the Navier-Stokes equations (NSE) in the wavelet basis [6, 10, 11, 12]. The application of the continuous wavelet transform with the derivatives of the Gaussian taken as basic wavelets was already used in the analytical study of the NSE [13, 14], in particular, in research on energy dissipation. However, at least to the authors knowledge *the wavelet decomposition has not been yet applied to the stochastic iterative solutions of the Navier-Stokes equation or in the framework of the field theory approach to statistical hydrodynamics* [15, 16, 17].

Our interest in extending the stochastic hydrodynamics approach by wavelet-defined random processes is stimulated by recent developments in application of the field theory and renormalization group methods to the fully developed isotropic turbulence, see e.g. [18] for a short summary. The "field theoretic methods" are taken to mean the stochastic diagram technique, the functional integral representation of the characteristic functional and the renormalization group methods. The latter, being inherited from the theory of critical behavior in equilibrium phase transitions clearly demonstrates the need for a scale-dependent random measure in the field theoretic approach to turbulence. Such measures, having been already in use for phenomenological studies of multifractal behavior and itermittency [19, 7, 8], have not been yet used in analytical approach of stochastic hydrodynamics.

The aim of this paper is to extend the wavelet representation of the stochastic Navier-Stokes equations in such a way that the probability distribution of the stirring force, used to compensate the energy dissipation, is defined for the wavelet coefficients of random force (*i.e.* for the scale components of forcing). In our description the velocity field wavelet coefficients $u_l(x)$ attain the Kolmogorov meaning of local velocity fluctuations of scale l at a given point x (A particular case of the difference of two Gaussians or delta-functions was considered in the literature [7, 20]). Defining the random force in the space of wavelet coefficients we get an extra analytical flexibility: there are random processes in the space of wavelet coefficients with different correlation functions whose images under inverse WT coincide in the space of common random functions. Tuning the random force correlation function in the space of wavelet coefficients, in stochastic hydrodynamics formalism, we get rid of loop divergences; for a special type of narrow-band forcing the contributions to the response and the correlation functions are explicitly calculated.

It is shown that the Kolmogorov hypotheses, the statistical closures of moment equations, the stochastic hydrodynamics approach and the Wyld diagram technique are naturally reformulated in multiscale (wavelet) formalism. Besides, the consideration of random processes depending on scale explicitly $\langle u_l(x)u_{l'}(x')\rangle = C(x,x',l,l')$ gives a possibility of the pertur-

bation expansion converging without introducing an ultra-violet (UV) cutoff wavenumber, and an extra contribution to the Kolmogorov energy dissipation term (u_l^3/l).

The rest of this paper is organized as follows. In *Section 2* we review the methods of solution of the NSE by the Fourier decomposition and by the wavelet decomposition. In *Section 3* the stochastic hydrodynamics approach is reformulated for the random processes explicitly depending on scale. A regularization of the perturbation expansion for a random force acting at a single scale is presented. Energy dissipation rate and energy flux in the multiscale formalism are considered in *Section 4*. *Section 5* presents a generalized form of the Kolmogorov hypotheses formulated in a multiscale framework. In *Conclusion* we discuss some perspectives of the method.

2 Hydrodynamics of incompressible fluid in wavelet representation

In analytical studies of the hydrodynamic turbulence the basic role is played by the Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla\left(\frac{p}{\rho}\right). \tag{1}$$

Our consideration of hydrodynamic turbulence, based on the Navier-Stokes equations, assumes a fully developed homogeneous isotropic turbulence far from any boundaries. Although the adequacy of the stochastic NSE to fully developed turbulence still remains an open problem, a significant progress has been achieved in studying simple models, proving the Kolmogorov spectrum from basic principles [21] and calculating the anomalous scaling corrections [22, 23, 24]. Turbulence in an incompressible fluid considered in these settings is more frequently studied in a wavenumber space rather than in a real space. The advantage of wavenumber space is a very simple form of the Laplacian – this enables to eliminate the pressure term from the NSE. The price paid for this simplification is the non-locality of the Fourier transform that completely hides all information related to the real space distribution of the velocity field. The discrete Fourier transform used in numerical simulations also imposes periodicity on the system. The usefulness of the wavenumber representation stems from two basic reasons: the existing of fast FFT algorithms and pseudospectral methods [25, 26] and the direct experimental interpretation of the power spectra density of velocity fluctuations [27]. For the incompressible fluid the pressure term can be eliminated from the NSE (1) by the substitution

$$p = -\frac{\rho}{\Delta}\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right). \tag{2}$$

In the wavenumber space representation

$$u_i(\mathbf{x},t) = \int e^{\imath \mathbf{kx}} \hat{u}_i(\mathbf{k},t) \frac{d^d\mathbf{k}}{(2\pi)^d}, \quad u_i(\mathbf{x},t) = \int e^{\imath(\mathbf{kx}-\omega t)} \hat{u}_i(\mathbf{k},\omega) \frac{d^{d+1}k}{(2\pi)^{d+1}}, \tag{3}$$

both the inverse Laplacian operator Δ^{-1} in (2) and the incompressibility condition $\nabla\cdot\mathbf{u}=0$ are simplified and the NSE system becomes a system of integro-differential equations

$$\begin{aligned} \left(\partial_t + \nu \mathbf{k}^2\right) \hat{u}_i(\mathbf{k},t) &= \int \frac{d^d\mathbf{q}}{(2\pi)^d} M_{ijk}(\mathbf{k}) \hat{u}_j(\mathbf{q},t) \hat{u}_k(\mathbf{k}-\mathbf{q},t), \\ \text{where} \quad M_{ijk}(\mathbf{k}) &= -\frac{\imath}{2}\left[k_j P_{ik}(\mathbf{k}) + k_k P_{ij}(\mathbf{k})\right], \quad P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}. \end{aligned} \tag{4}$$

The system of equations (4) allows for perturbative calculations, statistical closures of moment equations *etc.*., but is not local in the coordinate space and therefore is of little use when studying the effects locally produced by fluctuations. The system of equations (4) is complete and, being correctly solved numerically, gives reliable results. This however requires taking into account a tremendous number of Fourier modes, that is hardly bearable even for modern supercomputers; alternatively certain block-averaging procedures in wavenumber space can be applied, *e.g.* the spectral reduction [26].

The idea of studying turbulence using the space-scale or wavenumber-scale representation is not a new one. The band-pass filtering of the turbulent signals was used to study intermittency long ago [28]. Later the prototype of wavelet cascade model was given in [4]. The wavelet transform (WT) is known to be an excellent local tool, widely used in a turbulence data analysis [8] and numerical simulations [4, 6, 12]. WT in real space, as a particular type of filtering [29], reveals the formation of the coherent structures [8] and is useful to study the local energy dissipation effects related to filament formation [11]. WT in (a,k) space enables to enhance the idea of band-pass filtering by separating the contributions of different scales.

In this paper we use continuous wavelet transform (CWT) to derive the equations for the fluctuations of different scales. This analytical consideration goes along with the filtering approach [29, 30], when considering moment closures for a fully developed isotropic turbulence. *The crux of our approach is the application of continuous wavelet transform in the stochastic hydrodynamics framework: the WT is applied to both the velocity field and the random force stirring the turbulence.* Working in the space of wavelet coefficients, rather than in a real space of velocity fields, we can define the random stirring force, which is essentially used in RG calculations, see *e.g.* [31, 17] – in a way that allows for getting rid of loop divergences in the stochastic perturbation expansion of the velocity field statistical momenta [32].

For simplicity we restrict ourselves with the homogeneous isotropic turbulence and the isotropic wavelets $\psi(\mathbf{x}) = \psi(|\mathbf{x}|)$. In this case, the wavelet transform of the velocity field

$\mathbf{u}(x,t)$, taken with respect to the basic wavelet $\psi(\mathbf{x})$, and the corresponding reconstruction formula are

$$\mathbf{u}_a(\mathbf{b},t) = \int \frac{1}{|a|^d}\bar{\psi}\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right)\mathbf{u}(\mathbf{x},t)d^d\mathbf{x}, \tag{5}$$

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{C_\psi}\int \psi\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right)\mathbf{u}_a(\mathbf{b},t)\frac{da d^d\mathbf{b}}{|a|^{d+1}}. \tag{6}$$

We perform the wavelet transform only in the spatial argument of the velocity field because we need the *spatial* resolution. Using the L^1 norm instead of L^2, we provide wavelet coefficients $\mathbf{u}_a(\mathbf{b},t)$ with the same (LT^{-1}) dimension as the velocity field $\mathbf{u}(x)$ itself. The wavelet coefficients $\mathbf{u}_a(\mathbf{b},t)$ are referred to hereafter, as the components of the velocity field corresponding to scale a; $\psi(x)$ is referred to as an analyzing function used to measure the scale components.

For practical calculations it is often convenient to express wavelet transform (5,6) in $(a,\mathbf{k})$ representation taking the Fourier transform in the spatial argument. In Fourier form the direct and inverse WT are:

$$\hat{\mathbf{u}}_a(k) = \overline{\hat{\psi}(a\mathbf{k})}\hat{\mathbf{u}}(k), \quad \hat{\mathbf{u}}(k) = \frac{1}{C_\psi}\int \frac{da}{|a|}\hat{\psi}(a\mathbf{k})\hat{\mathbf{u}}_a(k), \tag{7}$$

where $\hat{\mathbf{u}}(k) \equiv \hat{\mathbf{u}}(\mathbf{k},\omega)$ is the Fourier transform of the velocity field

$$u(\mathbf{x},t) = \int e^{\imath(\mathbf{kx}-\omega t)}u(k)\frac{d^d\mathbf{k}d\omega}{(2\pi)^{d+1}}.$$

(The Minkovski-like notation $x \equiv (\mathbf{x},t), k \equiv (\mathbf{k},\omega)$ is used.) Therefore, the wavelet transform (5) can be considered as a frequency filter that conveys the harmonics with typical wavenumbers of order $\frac{1}{a}$ and is localized close to point $\mathbf{b}$.

The only restriction imposed on the basic wavelet ψ to make the wavelet transform invertible – the admissibility condition – is the finiteness of the normalization constant C_ψ:

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(a\mathbf{k})|^2}{|a|}da < \infty. \tag{8}$$

For a real-valued basic wavelet $\psi(\mathbf{x})$ we can restrict the integration to the positive frequencies only

$$C_\psi = 2\int_0^\infty \frac{|\hat{\psi}(a\mathbf{k})|^2}{a}da.$$

If the basic wavelet is also isotropic $\psi(\mathbf{x})=\psi(|\mathbf{x}|)$, we get

$$C_\psi=\int_{\mathbb{R}^d}\frac{|\hat\psi(\mathbf{k})|^2}{S_d|\mathbf{k}|^d}d^d\mathbf{k}, \tag{9}$$

where S_d is the area of the unit sphere in $\mathbb{R}^d$.

In this paper we will assume $a\in\mathbb{R}_+$ integration and the isotropic real wavelets. Thus, the decomposition of the velocity field with respect to the basic wavelet ψ takes the form

$$u(\mathbf{x},t) = \frac{2}{C_\psi}\int_0^\infty\frac{da}{a^{d+1}}\int_{\mathbb{R}^d}d^d\mathbf{b}\psi\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right)\mathbf{u}_a(\mathbf{b},t) \tag{10}$$

$$u(\mathbf{x},t) = \frac{2}{C_\psi}\int_0^\infty\frac{da}{a}\int_{\mathbb{R}^d}\frac{d^d\mathbf{k}d\omega}{(2\pi)^{d+1}}e^{\imath(\mathbf{kx}-\omega t)}\hat\psi(a\mathbf{k})\hat u_a(\mathbf{k},\omega). \tag{11}$$

We drop the integration limits $\int_0^\infty\frac{da}{a}$ hereafter.

Substituting the wavelet transform (11) into the system of the component equations (4), we yield the system of equations for the scale components $\hat{\mathbf{u}}_{ai}(k)$:

$$\begin{aligned}(-\imath\omega+\nu\mathbf{k}^2)\hat u_{ai}(k) &= \left(\frac{2}{C_\psi}\right)^2\int M_{ijk}^{aa_1a_2}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})\hat u_{a_1j}(q)\hat u_{a_2k}(k-q)\frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^{d+1}q}{(2\pi)^{d+1}}\\ M_{ijk}^{aa_1a_2}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q}) &= \overline{\hat\psi(a\mathbf{k})}M_{ijk}(\mathbf{k})\hat\psi(a_1\mathbf{q})\hat\psi(a_2(\mathbf{k}-\mathbf{q})).\end{aligned} \tag{12}$$

Let us derive statistical closures for the scale components. For this purpose we take Eq.(12) and its complex conjugate

$$\begin{aligned}(\partial_t+\nu\mathbf{k}^2)\hat u_{ai}(\mathbf{k},t) &= \left(\tfrac{2}{C_\psi}\right)^2\int M_{ijk}^{aa_1a_2}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})\hat u_{a_1j}(\mathbf{q},t)\hat u_{a_2k}(\mathbf{k}-\mathbf{q},t)\tfrac{da_1}{a_1}\tfrac{da_2}{a_2}\tfrac{d^d\mathbf{q}}{(2\pi)^d}\\ (\partial_{t'}+\nu\mathbf{k}^2)\overline{\hat u_{ai}(\mathbf{k},t')} &= \left(\tfrac{2}{C_\psi}\right)^2\int\overline{M_{ijk}^{aa_1a_2}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})\hat u_{a_1j}(\mathbf{q},t')\hat u_{a_2k}(\mathbf{k}-\mathbf{q},t')}\tfrac{da_1}{a_1}\tfrac{da_2}{a_2}\tfrac{d^d\mathbf{q}}{(2\pi)^d},\end{aligned} \tag{13}$$

multiply the first equation by $\overline{\hat u_{ai}(\mathbf{k},t')}$, sum up over the vector index i and take the statistical averaging $\langle\ \rangle$. Doing so, we get

$$\begin{aligned}(\partial_t+\nu\mathbf{k}^2)\sum_i\langle\overline{\hat u_{ai}(\mathbf{k},t')}u_{ai}(\mathbf{k},t)\rangle &= \left(\frac{2}{C_\psi}\right)^2\int\frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^d\mathbf{q}}{(2\pi)^d}M_{ijk}^{aa_1a_2}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})\\ &\quad\langle\overline{\hat u_{ai}(\mathbf{k},t')}\hat u_{a_1j}(\mathbf{q},t)\hat u_{a_2k}(\mathbf{k}-\mathbf{q},t)\rangle.\end{aligned}$$

Applying the same procedure to the second of the equations (13) and summing up the results

at coinciding time arguments $t\!=\!t'$, we get the moment equation

$$(\partial_t + 2\nu\mathbf{k}^2)\sum_i \langle\overline{\hat u_{ai}(\mathbf{k},t)}u_{ai}(\mathbf{k},t)\rangle = \left(\frac{2}{C_\psi}\right)^2 \int \frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^d\mathbf{q}}{(2\pi)^d} M^{aa_1a_2}_{ijk}(\mathbf{k},\mathbf{q},\mathbf{k}\!-\!\mathbf{q}) \langle\overline{\hat u_{ai}(\mathbf{k},t)}\hat u_{a_1j}(\mathbf{q},t)\hat u_{a_2k}(\mathbf{k}\!-\!\mathbf{q},t)\rangle + h.c., \qquad (14)$$

which is different from its plane wave counterpart only by extra scale indexes and extra integrations in scale logarithms $\int\frac{da}{a}$(octaves). To express the third order moments in (14) via the second moments, we must substitute

$$\hat u_{ai}(\mathbf{k},t) = \left(\frac{2}{C_\psi}\right)^2 \int_{-\infty}^t ds G^{aa_0}_{il}(\mathbf{k},t-s) M^{a_0a_1a_2}_{ljk}(\mathbf{k},\mathbf{q},\mathbf{k}\!-\!\mathbf{q}) \hat u_{a_1j}(\mathbf{q},s)\hat u_{a_2k}(\mathbf{k}\!-\!\mathbf{q},s)\frac{da_0}{a_0}\frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^d\mathbf{q}}{(2\pi)^d}, \qquad (15)$$

where $G^{aa_0}_{il}(\mathbf{k},t\!-\!s)$ is the response function. The difference from the standard plane-wave approach [5] is that additionally to the summation over vector indices we have to sum up over octaves to integrate over $\int\frac{da}{a}$ in each scale variable. So the statistical closures can be reproduced for the scale components.

In the zero-th order approximation (with no interaction term: $M(\cdot)\to 0$) the bare response function is given by

$$G^{[\mathrm{bare}]aa_0}_{il}(\mathbf{k},t\!-\!s) = \delta_{il}\int_{-\infty}^{\infty}\frac{\delta(a-a_0)a_0}{-\imath\omega+\nu\mathbf{k}^2}e^{-\imath\omega(t-s)}\frac{d\omega}{2\pi} = \delta_{il}\delta(a-a_0)a_0 e^{-\nu\mathbf{k}^2|t-s|}. \qquad (16)$$

The full response function, in view of the component equations (13), satisfies the integro-differential equation

$$(\partial_t\!+\!\nu\mathbf{k}^2)G^{aa_0}_{il}(\mathbf{k},t\!-\!s) = 2\left(\frac{2}{C_\psi}\right)^2\int M^{aa_1a_2}_{ijk}(\mathbf{k},\mathbf{q},\mathbf{k}\!-\!\mathbf{q})\hat u_{a_1j}(\mathbf{q},t)G^{a_2a_0}_{kl}(\mathbf{k}\!-\!\mathbf{q},t\!-\!s)\frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^d\mathbf{q}}{(2\pi)^d}.$$

The substitution of (15) into (14) gives a relation between the second and the forth order moments of the scale components:

$$(\partial_t + 2\nu\mathbf{k}^2)\sum_i \langle\overline{\hat u_{ai}(\mathbf{k},t)}\hat u_{ai}(\mathbf{k},t)\rangle = 2\left(\frac{2}{C_\psi}\right)^4\int\frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^d\mathbf{k}_1}{(2\pi)^d}\frac{da_3}{a_3}\frac{da_4}{a_4}\frac{d^d\mathbf{k}_2}{(2\pi)^d} \frac{da_0}{a_0}M^{aa_1a_2}_{ijk}(\mathbf{k},\mathbf{k}_1,\mathbf{k}\!-\!\mathbf{k}_1)\langle\overline{\hat u_{ai}(\mathbf{k},t)}\hat u_{a_1j}(\mathbf{k}_1,t)\int_{-\infty}^t ds G^{a_2a_0}_{kl}(\mathbf{k}\!-\!\mathbf{k}_1,t,s) M^{a_0a_3a_4}_{lrf}(\mathbf{k}\!-\!\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\hat u_{a_3r}(\mathbf{k}_2,s)\hat u_{a_4f}(\mathbf{k}\!-\!\mathbf{k}_1\!-\!\mathbf{k}_2,s)\rangle + h.c.. \qquad (17)$$

The forth order moments $\langle uuuu\rangle$ can be further decomposed into the sum of all pairs $\langle uu\rangle\langle uu\rangle$ using a stochastic perturbation expansion.

3 Stochastic hydrodynamics with multiscale forcing

The stochastic hydrodynamics approach consists in introducing random force in the Navier-Stokes equations and calculating the velocity field momenta $\langle \mathbf{u}(x_1)\ldots\mathbf{u}(x_n)\rangle$ using the stochastic perturbation theory, pioneered by Wyld [33], or the functional integral formalism.

In a coordinate representation, the stochastic NSE is written in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla\left(\frac{p}{\rho}\right) + \eta(\mathbf{x},t). \tag{18}$$

The random force correlator $\langle \eta_i(x)\eta_j(x')\rangle = D_{ij}(x-x')$ should obey certain conditions to make the resulting theory physically feasible and the perturbation expansion suitable for analytical evaluation. First, the energy injection by random force should be equal to the energy dissipation; secondly, the forcing should be essentially infrared (IR), *i.e.* it should be localized at large scales; third, it is desirable to have a parameter to control the IR divergences (when the size of the system tends to infinity).

To exclude the pressure term in (18) using the incompressibility condition, the Fourier representation is used

$$\left(-\imath\omega + \nu\mathbf{k}^2\right)\hat{u}_i(\mathbf{k},\omega) - \int \frac{d^{d+1}q}{(2\pi)^{d+1}} M_{ijk}(\mathbf{k})\hat{u}_j(q)\hat{u}_k(k-q) = \hat{\eta}_i(k). \tag{19}$$

The random force correlator is usually taken in the form

$$\langle \hat{\eta}_i(k_1)\hat{\eta}_j(k_2)\rangle = (2\pi)^{d+1}\delta^{d+1}(k_1+k_2)P_{ij}(\mathbf{k}_1)D(\mathbf{k}_1), \tag{20}$$

where the function $D(|\mathbf{k}|)$ has a suitable power-law behavior. In the simplest, but not very feasible physically, case $D(\mathbf{k}) = D_0 = const$, we deal with the white noise, δ-correlated in both space and time.

What is most realistic physically, is to have a random force concentrated in a limited domain in k-space $\Lambda_{min} < |\mathbf{k}| < \Lambda_{max}$. This case, however, is difficult to evaluate analytically in the perturbation theory [15]. In the multiscale approach we are going to present, we solve this problem by constructing a noise acting in a limited domain of scales a in $(a,\mathbf{k})$ space.

In the $(a,\mathbf{k})$ representation, having excluded the pressure by standard means of the orthogonal projector, the Eq. (18) leads to a system of integro-differential equations for the scale components $\hat{u}_{ai}(k)$:

$$\begin{aligned}(-\imath\omega + \nu\mathbf{k}^2)\hat{u}_{ai}(k) &= \hat{\eta}_{ai}(k) \\ &+ \left(\frac{2}{C_\psi}\right)^2 \int M^{aa_1a_2}_{ijk}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})\hat{u}_{a_1j}(q)\hat{u}_{a_2k}(k-q)\frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^{d+1}q}{(2\pi)^{d+1}}.\end{aligned} \tag{21}$$

Now we face the problem of appropriate choice of the force correlator

$$\langle \hat\eta_{ia_1}(k_1)\hat\eta_{ja_2}(k_1)\rangle = D_{ij}^{a_1a_2}(k_1,k_2).$$

It was shown in the previous paper [32] that the (a,k) representation provides an extra analytical flexibility in constructing random processes with desired correlation properties in a coordinate space. For instance, the random process given by wavelet coefficients with the correlation function

$$\langle \hat\eta_{a_1}(\mathbf{k}_1)\hat\eta_{a_2}(\mathbf{k}_2)\rangle = (2\pi)^d \frac{C_\psi}{2}\delta^d(\mathbf{k}_1+\mathbf{k}_2)a_1\delta(a_1-a_2)D_0 \tag{22}$$

possesses the same correlation properties in $\mathbb{R}^d$ coordinate space as the white noise does. Casting $\hat\eta(k)$ in terms of $\hat\eta_a(k)$ by means of (7) we get

$$\begin{aligned}
\langle \hat\eta(\mathbf{k}_1)\hat\eta(\mathbf{k}_2)\rangle &= \left(\frac{2}{C_\psi}\right)^2 \int \frac{da_1}{a_1}\frac{da_2}{a_2}\hat\psi(a_1\mathbf{k}_1)\hat\psi(a_2\mathbf{k}_2)\langle \hat\eta_{a_1}(\mathbf{k}_1)\hat\eta_{a_2}(\mathbf{k}_2)\rangle \\
&= (2\pi)^d \frac{2D_0}{C_\psi}\delta^d(\mathbf{k}_1+\mathbf{k}_2)\int \frac{da_1}{a_1}\frac{da_2}{a_2}\hat\psi(a_1\mathbf{k}_1)\hat\psi(a_2\mathbf{k}_2)a_1\delta(a_1-a_2) \\
&= (2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2)D_0
\end{aligned}$$

that coincides with the correlation function of the white noise. The direct wavelet transform of the white noise $\eta(\mathbf{x})\to\hat\eta(\mathbf{k})\to\hat\eta_a(\mathbf{k})$ apparently leads to another result

$$\langle \hat\eta_{a_1}(\mathbf{k}_1)\hat\eta_{a_2}(\mathbf{k}_2)\rangle = (2\pi)^d\delta^d(\mathbf{k}_1+\mathbf{k}_2)D_0\overline{\hat\psi(a_1\mathbf{k}_1)\hat\psi(a_2\mathbf{k}_2)}, \tag{23}$$

which is different from (22) and explicitly depends on the basic wavelet ψ.

Physically, the scale-dependent processes obeying (22) and (23), respectively, describe quite different processes: fluctuations of the former type (22) are mutually correlated only for coinciding scales $(a_1=a_2)$, while for the latter case (23) all fluctuations are correlated.

Exactly as in the standard wavenumber space approach [15, 16, 17], we can generalize the δ-correlated force (22) assuming its variance to be dependent on both the scale the wave vector: $D_0 \to D(a,\mathbf{k})$. Taking into account we deal with the incompressible fluid in d dimensions, we can put down a general form of the desired force correlator

$$\langle \hat\eta_{a_1i}(k_1)\hat\eta_{a_2j}(k_2)\rangle = (2\pi)^{d+1}\delta^{d+1}(k_1+k_2)\frac{C_\psi}{2}a_1\delta(a_1-a_2)P_{ij}(\mathbf{k}_1)D(a_1,|\mathbf{k}_1|). \tag{24}$$

The δ-correlated random force in the wavenumber space does not provide an adequate description of hydrodynamic turbulence, for it gives an energy injection in all scales, small

and large. In physical settings, the fluid is usually stirred at a predetermined scale, or in a narrow range of scales, comparable to the size of the system. As a simplest model of such a forcing, we can consider a force acting on a single scale a_0 by choosing

$$D(a,\mathbf{k}) = D_0 a_0 \delta(a-a_0). \tag{25}$$

Now let us turn to the perturbative calculations. The stochastic diagram techniques for the component fields $\hat{\mathbf{u}}_a(k)$ stems from Eq. (21) and is a straightforward generalization of the Wyld diagram technique for the Fourier components $\hat{\mathbf{u}}(\mathbf{k})$:

$$\begin{aligned}\hat{u}_{ai}(k) &= G_0(k)\hat{\eta}_{ai}(k) + G_0(k)\left(\frac{2}{C_\psi}\right)^2 \int \frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^{d+1}q}{(2\pi)^{d+1}} \\ & M^{aa_1a_2}_{ijk}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})\hat{u}_{a_1j}(q)\hat{u}_{a_2k}(k-q),\end{aligned} \tag{26}$$

where $G_0(k) = (-\imath\omega + \nu\mathbf{k}^2)^{-1}$ is the bare response function for Fourier component. To keep the scales and wavevectors on the same footing and make the notation covariant in that sense, we can rewrite (26) using the response functions bearing scale indices explicitly (16). Thus, we get

$$\begin{aligned}\hat{u}_{ai}(k) &= \int \frac{da_0}{a_0} G^{aa_0}_{0ij}(k)\hat{\eta}_{a_0j}(k) + \left(\frac{2}{C_\psi}\right)^2 \int \frac{da_0}{a_0} G^{aa_0}_{0il}(k)\frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^{d+1}q}{(2\pi)^{d+1}} \\ & M^{aa_1a_2}_{ljk}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})\hat{u}_{a_1j}(q)\hat{u}_{a_2k}(k-q) \\ G^{aa_0}_{0ij}(k) &= \frac{\delta(a-a_0)a_0}{-\imath\omega+\nu\mathbf{k}^2}\delta_{ij}.\end{aligned} \tag{27}$$

The Feynman expansion for the scale component fields $\hat{u}_{ai}(k)$ can be derived either from (26) or (27). Iterating the Eq. (26) once, we get the one-loop contribution to the response function:

$$\begin{aligned}\hat{u}_{ai}(k) &= G_0(k)\hat{\eta}_{ai}(k) + G_0(k)\left(\frac{2}{C_\psi}\right)^2 \int \frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^{d+1}k_1}{(2\pi)^{d+1}} \\ & M^{aa_1a_2}_{ijk}(\mathbf{k},\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1)\hat{u}_{a_1j}(k_1)\Big[G_0(k-k_1)\hat{\eta}_{a_2k}(k-k_1) \\ &+ G_0(k-k_1)\left(\frac{2}{C_\psi}\right)^2 \int \frac{da_3}{a_3}\frac{da_4}{a_4}\frac{d^{d+1}k_2}{(2\pi)^{d+1}} M^{a_2a_3a_4}_{klm}(\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) \\ & \hat{u}_{a_3l}(k_2)\hat{u}_{a_4m}(k-k_1-k_2)\Big].\end{aligned} \tag{28}$$

As usual, we assume the random force to be gaussian, with all odd correlators $\langle \eta_1 \ldots \eta_{2k+1} \rangle$ vanish identically.

Following [15], we introduce a formal parameter of the perturbation expansion λ in the interaction term ($M_{ijk}^{aa_1a_2} \to \lambda M_{ijk}^{aa_1a_2}$). In the final end the initial value $\lambda = 1$ should be restored. The validity of considering λ as a small parameter of perturbation expansion is justified by renormalization group methods [15, 16, 17]; see also [24] for recent developments and generalizations.

In the zero-th order of perturbation expansion the response function does not depend on scale and coincides with G_0 for all scale components:

$$\hat{u}_{ai}(k) = G_0(k)\hat{\eta}_{ai}(k). \tag{29}$$

In the $O(\lambda^2)$ and the next orders of perturbation expansion the standard stochastic diagram techniques, used by many authors [33, 15, 17], is reproduced with the difference that: (i) each vertex, each response and correlation function attain scale superscripts; (ii) integration over octaves $\int \frac{da}{a}$ is performed over all pair-matching scale indices. Mathematically, this means that each diagram line corresponding to the plane wave component $\hat{u}_i(k)$ in standard techniques now attains an extra wavelet factor and becomes $\hat{u}_{ai}(k) = \overline{\hat{\psi}(a\mathbf{k})\hat{u}_i(k)}$. The Feynman graphs and their topological factors of course remain the same.

Here, for bookkeeping reasons, we present only one loop 1PI contributions to the response, see Fig. 1, and correlation, see Fig. 2, functions of stochastic hydrodynamics. In the first order in the force correlator $\langle\eta\eta\rangle$ (one-loop contribution) we substitute the scale components $\hat{u}$ in the r.h.s. of (28) by the zero-th order solutions (29), perform necessary scale integrations in $a_l\delta(a-a_l)\frac{da_l}{a_l}$ and average over the random force (24):

$$\begin{aligned}
\hat{u}_{ai}(k) \;=\;& G_0(k)\hat{\eta}_{ai}(k) + G_0(k)4\lambda^2\left(\frac{2}{C_\psi}\right)^4 \int \frac{da_1}{a_1}\frac{da_2}{a_2}\frac{da_3}{a_3}\frac{da_4}{a_4}\frac{d^{d+1}k_1}{(2\pi)^{d+1}}\frac{d^{d+1}k_2}{(2\pi)^{d+1}} \\
& M_{ijk}^{aa_1a_2}(\mathbf{k},\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1)G_0(k_1)\langle\hat{\eta}_{a_1j}(k_1)\hat{\eta}_{a_3l}(k_2)\rangle G_0(k_2)G_0(k-k_1) \\
& M_{klm}^{a_2a_3a_4}(\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)G_0(k-k_1-k_2)\hat{\eta}_{a_4m}(k-k_1-k_2).
\end{aligned}$$

The corresponding diagram is shown in Fig. 1. The factor 4 accounts for two possible ways to expand the nonlinear term multiplied by two possible ways of taking the random force averaging. After substituting the random force correlator (24), the using of the explicit form of the interaction vertexes (12), and integrating over the scales, this leads to

$$\begin{aligned}
\hat{u}_{ai}(k) \;=\;& G_0(k)\hat{\eta}_{ai}(k) + \overline{\hat{\psi}(a\mathbf{k})}G_0^2(k)4\lambda^2\left(\frac{2}{C_\psi}\right)\int \frac{d^{d+1}k_1}{(2\pi)^{d+1}}\frac{da_4}{a_4}M_{ijk}(\mathbf{k}) \qquad (30)\\
& |G_0(k_1)|^2\Delta_{jl}(\mathbf{k}_1)G_0(k-k_1)M_{klm}(\mathbf{k}-\mathbf{k}_1)\hat{\psi}(a_4\mathbf{k})\hat{\eta}_{a_4m}(k), \\
\Delta_{jl}(\mathbf{k}_1) \;=\;& P_{jl}(\mathbf{k}_1)\left(\frac{2}{C_\psi}\right)\int \frac{da_1}{a_1}|\hat{\psi}(a_1\mathbf{k}_1)|^2 D(a_1,\mathbf{k}_1). \qquad (31)
\end{aligned}$$

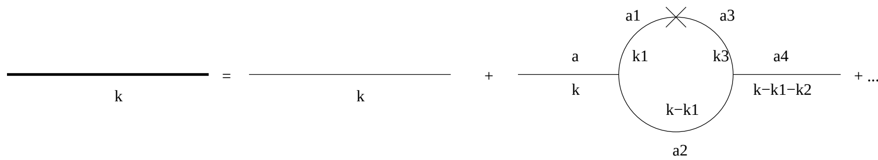

Figure 1: One loop contribution to the response function

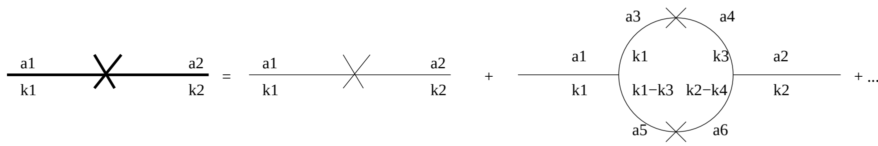

Figure 2: One loop contribution to the correlation function

Eq.(31) means that for our special type of the scale-dependent random forcing (24) all internal parts of the diagrams, that do not carry the scale indices, can be evaluated by substitution of the effective force correlator (31) into the standard diagrams drawn in wavenumber space. In this way we can easily evaluate the perturbative corrections to the usual response function $G(k)$ for Fourier components (see Appendix), and hence evaluate the turbulent corrections to viscosity, produced by scale-dependent force.

Similarly, we can evaluate the contributions to the correlation functions $\langle \hat{u}_{a_1 i}(k_1)\hat{u}_{a_2 j}(k_2)\rangle$. In the one-loop approximation, using (28) and the zero-th order approximation (29), we get

$$\begin{aligned}
\langle \hat{u}_{a_1 i}(k_1)\hat{u}_{a_2 j}(k_2)\rangle &= G_0(k_1)G_0(k_2)\langle \hat{\eta}_{a_1 i}(k_1)\hat{\eta}_{a_2 j}(k_2)\rangle \\
&+ 2\lambda^2 G_0(k_1)G_0(k_2)\left(\frac{2}{C_\psi}\right)^4 \int \frac{da_3}{a_3}\frac{da_5}{a_5}\frac{da_4}{a_4}\frac{da_6}{a_6}\frac{d^{d+1}k_3}{(2\pi)^{d+1}}\frac{d^{d+1}k_4}{(2\pi)^{d+1}} \\
&\quad M^{a_1 a_3 a_5}_{ilk}(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3)G_0(k_3)\langle \hat{\eta}_{a_3 l}(k_3)\hat{\eta}_{a_4 m}(k_4)\rangle G_0(k_4)G_0(k_1-k_3) \\
&\quad \langle \hat{\eta}_{a_5 k}(k_1-k_3)\hat{\eta}_{a_6 n}(k_2-k_4)\rangle G_0(k_2-k_4)M^{a_2 a_4 a_6}_{jmn}(\mathbf{k}_2,\mathbf{k}_4,\mathbf{k}_2-\mathbf{k}_4).
\end{aligned}$$

After the integrations over a_4, a_6, k_4, identically to the response functions calculations, with the random force (24), we get the one-loop contribution to the correlation function, shown in Fig. 2:

$$\begin{aligned}
\langle \hat{u}_{a_1 i}(k_1)\hat{u}_{a_2 j}(k_2)\rangle &= |G_0(k_1)|^2\langle \hat{\eta}_{a_1 i}(k_1)\hat{\eta}_{a_2 j}(k_2)\rangle + 2\lambda^2\delta^{d+1}(k_1+k_2)\left(\frac{2}{C_\psi}\right)^2 \int \frac{da_3}{a_3}\frac{da_5}{a_5}\frac{d^{d+1}k_3}{(2\pi)^{d+1}} \\
&\quad M^{a_1 a_3 a_5}_{ilk}(\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3)|G_0(k_3)|^2 P_{lm}(\mathbf{k}_3)D(a_3,\mathbf{k}_3)|G_0(k_1-k_3)|^2 \\
&\quad P_{kn}(\mathbf{k}_1-\mathbf{k}_3)D(a_5,\mathbf{k}_1-\mathbf{k}_3)M^{a_2 a_4 a_6}_{jmn}(-\mathbf{k}_1,-\mathbf{k}_3,-\mathbf{k}_1+\mathbf{k}_3). \qquad (32)
\end{aligned}$$

Expanding the wavelet factors $\psi(a\mathbf{k})$ in each vertex and integrating over all matching scale arguments, we get the one-loop correction to the correlation function

$$\begin{aligned} C_2(a_1,k_1,a_2,k_2) &= 2\delta^{d+1}(k_1+k_2)\lambda^2|G_0(k_1)|^2\overline{\hat\psi(a_1\mathbf{k}_1)\hat\psi(-a_2\mathbf{k}_1)}\int\frac{d^{d+1}k_3}{(2\pi)^{d+1}}M_{ilk}(\mathbf{k}_1) \\ & |G_0(k_3)|^2\Delta_{lm}(\mathbf{k}_3)|G_0(k_1-k_3)|^2\Delta_{kn}(\mathbf{k}_1-\mathbf{k}_3)M_{jmn}(-\mathbf{k}_1). \end{aligned} \tag{33}$$

The evaluation of the one-loop contribution to the correlation function is easily performed for the above mentioned narrow-band force correlators. The contribution takes the form

$$C_2(a_1,k_1,a_2,k_2) = \delta^{d+1}(k_1+k_2)\hat\psi(a_1\mathbf{k}_1)\psi(-a_2\mathbf{k}_2)C_{eff}(k_1)$$

$$C_{eff}(k) = 2\lambda^2|G_0(k)|^2\int\frac{d^{d+1}q}{(2\pi)^{d+1}}\Delta(\mathbf{q})\Delta(\mathbf{k}-\mathbf{q})c_2(\mathbf{k},\mathbf{q})|G_0(q)|^2|G_0(k-q)|^2, \tag{34}$$

where the trace of the one-loop tensor structure $c_2(\mathbf{k},\mathbf{q})$ is given in Appendix. The integration in frequency argument in the limit of zero frequency ($k_0\to0$), is not different from that in stochastic hydrodynamics in the wavenumber space and gives

$$\begin{aligned} C_{eff}(\mathbf{k}) &= \lambda^2|G_0(0,\mathbf{k})|^2S_{d-1}\int\frac{q^{d-1}dqd\theta\sin^{d-2}\theta\Delta(\mathbf{q})\Delta(\mathbf{k}-\mathbf{q})}{\nu^3q^2(k^2-2kq\cos\theta+q^2)(k^2-2kq\cos\theta+2q^2)} \\ & \frac{(1-\cos^2\theta)k^2}{4(k^2-2kq\cos\theta+q^2)}\left[k^2(d-1)-2kqd\cos\theta+2q^2(d+2\cos^2\theta-2)\right]. \end{aligned} \tag{35}$$

As an example, let us present the one-loop contribution to the effective pair correlator (34), calculated for the case of single-scale forcing (25) with the basic wavelets from the gaussian vanishing momenta wavelet family

$$\hat g_n(k) = (2\pi)^{\frac{d}{2}}(-\imath\mathbf{k})^n\exp(-\mathbf{k}^2/2), \quad C_{g_n} = (2\pi)^d\Gamma(n), \tag{36}$$

with $\Gamma(x)$ being the Eulerian gamma-function.

For the single-scale forcing (25) this gives an effective force correlator in the wavenumber space

$$\Delta_n(q) = \frac{D_0}{\Gamma(n)}(a_0q)^{2n}e^{-(a_0q)^2}. \tag{37}$$

Straightforward calculation leads to:

$$\begin{aligned} C_{eff}(k) &= \lambda^2\frac{k^2|G_0(k)|^2S_{d-1}a_0^{4n}D_0^2}{4\nu^3\Gamma(n)^2}\int\frac{q^{2n}(k^2-2kq\cos\theta+q^2)^ne^{-a_0^2(k^2-2kq\cos\theta+2q^2)}}{q^2(k^2-2kq\cos\theta+q^2)^2(k^2-2kq\cos\theta+2q^2)} \\ & (1-\cos^2\theta)\left[k^2(d-1)-2kqd\cos\theta+2q^2(d+2\cos^2\theta-2)\right]q^{d-1}dq\sin^{d-2}\theta d\theta. \end{aligned} \tag{38}$$

The integration over the angle variable $\cos\theta$ can be performed explicitly. With the calculations presented in Appendix, we get the effective pair correlator for $n = 2, d = 3$ in the large-scale limit $\left(x = \frac{k}{q} \to 0\right)$:

$$C_{eff}^{d=3,n=2}(k \to 0) = \frac{7}{40} \frac{k^2 |G_0(k)|^2 \pi^{\frac{3}{2}} a_0^3 D_0^2}{\nu^3 \sqrt{2}} \lambda^2. \tag{39}$$

4 Energy dissipation and energy transfer

The energy dissipation rate per unit of mass of an incompressible viscous fluid is given by the Navier-Stokes equations

$$\epsilon = -\nu \int u(x) \Delta u(x) d^d\mathbf{x} = \nu \int d^d\mathbf{x} (\nabla u)^2.$$

Using wavelet decomposition (6) for velocity field we get

$$\epsilon = -\nu \int \Omega(a_1, a_2, \mathbf{b}_1 - \mathbf{b}_2) u_{a_1}(b_1) u_{a_2}(b_2) \frac{da_1 d^d\mathbf{b}_1}{a_1} \frac{da_2 d^d\mathbf{b}_2}{a_2}, \tag{40}$$

where

$$\begin{aligned} \Omega(a_1, a_2, \mathbf{b}_1 - \mathbf{b}_2) &= \left(\frac{2}{C_\psi}\right)^2 \int \frac{d^d\mathbf{x}}{(a_1 a_2)^d} \psi\left(\frac{\mathbf{x} - \mathbf{b}_1}{a_1}\right) \frac{\partial^2}{\partial \mathbf{x}^2} \psi\left(\frac{\mathbf{x} - \mathbf{b}_2}{a_2}\right) \\ &= -\left(\frac{2}{C_\psi}\right)^2 \int \mathbf{k}^2 \hat{\psi}(a_1\mathbf{k}) \hat{\psi}(-a_2\mathbf{k}) e^{-\imath\mathbf{k}(\mathbf{b}_1 - \mathbf{b}_2)} \frac{d^d\mathbf{k}}{(2\pi)^d} \end{aligned}$$

is the dissipation connection for the scale components.

In symbolic form, the contribution of the fluctuations of all scales a_i to the mean dissipation of energy per unit of mass can be written as

$$\epsilon = \sum_{ij} \nu_{ij} \int u_{a_i}(x_i) u_{a_j}(x_j) d^d\mathbf{x}_i d^d\mathbf{x}_j, \tag{41}$$

where ν_{ij} is the viscosity between a_i and a_j scales. Being well localized in both the real and the wavenumber space, the analyzing function ψ perceives the interaction of the components of the same or close scales stronger than the contributions of the significantly different ones $|\log(a_1/a_2)| \gg 1$. For the Daubechies wavelets (the orthogonal wavelets with compact support) often used for numerical simulation of turbulence, the viscosity connection coefficients can be found elsewhere [34].

For a qualitative estimation of the behavior of the viscosity connection as a function of the scale ratio a_1/a_2 of the interacting scale components, let us consider the vanishing momenta wavelet family of gaussian wavelets (36), considered by Lewalle in a wavelet-based analysis of energy dissipation [13, 14]. The viscosity connection (41) can be then evaluated analytically:

$$\begin{aligned} \Omega_n(a_1,a_2,b_1-b_2) &= -\frac{(a_1a_2)^n}{\Gamma^2(n)}\int_{-\infty}^{\infty}(k^2)^{n+1}e^{-\frac{k^2}{2}(a_1^2+a_2^2)-\imath(b_1-b_2)k}\frac{d^dk}{(2\pi)^d} \\ &= \frac{(a_1a_2)^n}{\Gamma^2(n)}(-1)^n2^{n+1}\left.\frac{d^{n+1}}{d\sigma^{n+1}}\frac{e^{-\frac{(b_1-b_2)^2}{2\sigma}}}{\sigma^{d/2}}\right|_{\sigma=a_1^2+a_2^2} . \end{aligned} \tag{42}$$

The main contribution to energy dissipation comes from the terms with coinciding or closed arguments $x=b_1-b_2\approx 0$. In this limit (with $d=1$, taken for simplicity) we get

$$\Omega_n \sim \frac{1}{a_1^2+a_2^2}\left(\frac{a_1a_2}{a_1^2+a_2^2}\right)^{\frac{2n+1}{2}}(2n+1)!!.$$

Introducing the ratio $t=\frac{a_1}{a_2}$, we can study the behavior of the viscosity connection as a function of scale ratio

$$\Omega_n \sim \frac{1}{a_2^2(1+t^2)}\left(\frac{t}{t^2+1}\right)^{\frac{2n+1}{2}}(2n+1)!!.$$

The plot of the viscosity connection $\Omega_n(a_1,a_2,0)$ as a function of the scale ratio $t=\frac{a_1}{a_2}$, for the first three wavelets ($n=1,2,3$) of the (36) family, is shown in Fig. 3. As it can be seen, regardless the number of vanishing momenta n, the dissipation term has a maximum at coinciding scales ($t\approx 1$). For this reason, if a discrete wavelet decomposition is used instead of continuous one, it is sufficient to keep the two main terms in the energy dissipation: the equal-scale interaction and the neighboring scale interaction

$$\epsilon \sim -u_k^ju_m^j\int dx\psi_k^j(x)\Delta\psi_m^j(x) - 2u_k^{j-1}u_m^j\int dx\psi_k^{j-1}(x)\Delta\psi_m^j(x)+\ldots. \tag{43}$$

The first term is the standard viscosity term, the last is the Kraichnan nearest scales interaction.

The nonlinear energy transfer between neighboring scales can be evaluated by considering the wavelet connections corresponding to nonlinear term $(u\nabla)u$ of the Navier-Stokes

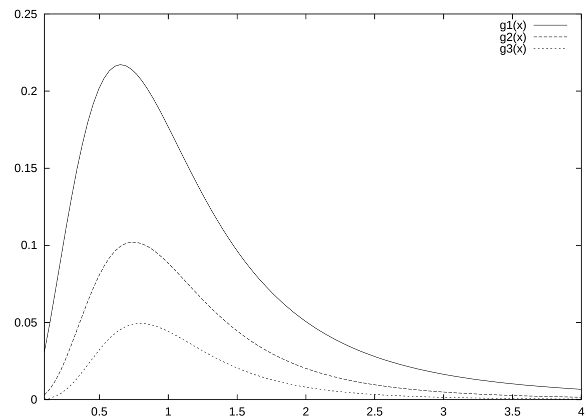

Figure 3: Graph of the function $\frac{1}{a_2^2(1+t^2)}\left(\frac{t}{t^2+1}\right)^{\frac{2n+1}{2}}$ for $n = 1, 2, 3$ related to the viscosity connection for gaussian type wavelets.

equation. To keep with the wavelet turbulence cascade models [4], we restrict ourselves with a discrete wavelet transform with binary scale step, $a_0 = \frac{1}{2}$:

$$\mathbf{u}(x) = \sum_{j\mathbf{k}} \mathbf{u}_{\mathbf{k}}^j \psi_{\mathbf{k}}^j(\mathbf{x}) + \text{Error term}, \quad \psi_{\mathbf{k}}^j(\mathbf{x}) \equiv a_0^{-\frac{jd}{2}} \psi\left(\frac{\mathbf{x} - \mathbf{k} b_0 a_0^j}{b_0 a_0^j}\right). \tag{44}$$

Without loss of generality the mesh size is set to unity $b_0 = 1$. In our consideration of the DWT representation (44) applied to hydrodynamic turbulence, in contrast to many schemes applied for numerical simulation of the NSE [35, 10], we have no *a priory* arguments to assume a mutual orthogonality of the basis functions ψ_k^j. To keep the wavelet decomposition (44) unique, the orthogonality of basic functions is not required. It is sufficient if the set of basic functions forms a *frame*, *i.e.* $\forall f \in L^2(\mathbb{R}), \exists A > 0, B < \infty$ such that

$$A\|f\|^2 \le \sum_{jk} |\langle f|\psi_k^j\rangle|^2 \le B\|f\|^2.$$

If $A = B$, the frame is called a *tight frame*.

Assuming the basic functions ψ_k^j form a frame, and restricting ourselves with the case of incompressible fluid, we cast the NSE system in the form

$$\frac{\partial u_k^{j\alpha}}{\partial t}\psi_k^j(x) + u_m^{l\beta}\psi_m^l(x)\frac{\partial u_k^{j\alpha}\psi_k^j(x)}{\partial x^\beta} = -\frac{1}{\rho}\frac{\partial p_k^j \psi_k^j(x)}{\partial x^\alpha} + \nu u_k^{j\alpha}\Delta\psi_k^j(x), \tag{45}$$

with Greek letters used for the coordinate indices, the bold face for the vector subscripts being dropped, and the summation over all pair-matching indices assumed. The component fields $u_k^{j\alpha} = u_k^{j\alpha}(t)$ are the functions of time only; so we deal with a typical cascade model.

Our goal at this point is to derive the energy transfer between the components of j-th scale and the next small $(j+1)$-th one. Let us define the energy of the j-th scale pulsations as

$$E_j = \frac{1}{2}\sum_{\alpha k} \bar{u}_k^{j\alpha} u_m^{j\alpha} \Lambda_{km}^j, \quad \Lambda_{km}^j = \int d^d x \bar{\psi}_k^j(x)\psi_m^j(x), \tag{46}$$

with the assumed unit normalization of the basic function $\int d^d x |\psi(x)|^2 = 1$. The contribution of the nonlinear term of the NSE to the time derivative of E_j is

$$\Delta E_j = -\Delta t \bar{u}_s^{r\alpha} u_m^{l\beta} u_k^{j\alpha} \int d^d x \bar{\psi}_s^r(x)\psi_m^l(x)\frac{\partial \psi_k^j(x)}{\partial x^\beta}. \tag{47}$$

For the orthogonal wavelets, that are most often used in numerical simulations [35], only the terms of coinciding scales $r = l = j$ survive in the r.h.s. of (47). The energy flux from the j-th scale to the next $(j+1)$-th scale is then proportional to $|u^j|^3/(b_0 a_0^j)$, in exact accordance to the Kolmogorov phenomenological theory [1]. In more general terms of nonorthogonal basic functions, the next term in the r.h.s. of (47) is proportional to $u^{j+1}u^{j+1}u^j$. This term can be interpreted as the material derivative $u^j \nabla (u^{j+1})^2$ of the mean energy $\frac{(u^{j+1})^2}{2}$ of the small scale fluctuations travelling along the stream of large scale velocity u^j.

The energy transfer terms analogues to (47) have been already considered in the orthogonal wavelet formalism by C.Meneveau [6]. They can be obtained directly from the component equations, by multiplying (21) by $\overline{\hat{u}_{ai}}(k)$. This leads to the energy transfer in $(a, \mathbf{k})$ space

$$t(a,k) = \left(\frac{2}{C_\psi}\right)^2 \int \overline{\hat{u}_{ai}}(k) M_{ijk}^{aa_1a_2}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})\hat{u}_{a_1j}(q)\hat{u}_{a_2k}(k-q)\frac{da_1}{a_1}\frac{da_2}{a_2}\frac{d^{d+1}q}{(2\pi)^{d+1}}, \tag{48}$$

given in [6].

5 Kolmogorov hypotheses

The Kolmogorov theory of the locally isotropic turbulence is formulated in terms of relative velocities

$$\delta u(r,l) = u(r+l) - u(r). \tag{49}$$

The probability distribution of relative velocities (49) hardly can be studied by the Fourier transform, in case the velocity field $u(r)$ is not homogeneous. According to Kolmogorov

[1], the turbulence in space-time domain G, is referred to as a stationary turbulence if, for any fixed $u(r,t)$, the distribution of relative velocities $\delta u(r,t)$ is stationary and isotropic. Physical assumptions on the locally isotropic turbulence were formulated in terms of the first and second Kolmogorov hypotheses, that reside on the definition of the Reynolds number. This is not a quite rigorous mathematical definition. We shall show that the Kolmogorov hypotheses are the statements about the behavior of the scale components of velocity field.

First, we have to note that the definition of the Reynolds number is consistent within the multiscale framework. In fact, by definition

$$Re_l = \frac{u_l l}{\nu}, \tag{50}$$

where u_l are said to be "pulsations of the scale l", the rigorous definition of those can be given by virtue of wavelet components (5), considering ψ as an apparatus function used to measure the pulsations. Going further, we find the wavelet components (5) to be identical to velocity increments (49), in case ψ is the Haar wavelet:

$$h(x) = \begin{cases} 1 & 0 \le x < 1/2 \\ -1 & 1/2 \le x < 1 \\ 0 & \text{otherwise} \end{cases} . \tag{51}$$

Now let us consider the Kolmogorov hypotheses [1]:

H1: The first hypothesis of similarity For the locally isotropic turbulence with high enough Re the PDFs for the relative velocities (49) are uniquely determined by the viscosity ν and the mean energy dissipation rate ϵ.

H2: The second hypothesis of similarity Under the same assumption as for H1 the turbulent flow is self-similar in small (but still $l \gg \nu^{\frac{3}{4}}\epsilon^{-\frac{1}{4}}$) scales in the sense that

$$\delta u(r, \lambda l) \stackrel{law}{=} \lambda^h \delta u(r,l), \quad \lambda \in \mathbb{R}_+. \tag{52}$$

Using the Taylor frozen flow hypothesis, and therefore considering one-dimensional pulsations $u(x)$, according to the definition of wavelet coefficients (5), with $\psi = h$ we get

$$\begin{aligned} u_l(r) &= \int_{-\infty}^{\infty} \frac{1}{l} h\left(\frac{x-r}{l}\right) u(x)dx = \int_0^1 dt h(t) u(lt+r) \\ &= \int_0^{1/2} u(lt+r)dt - \int_{1/2}^1 u(lt+r)dt. \end{aligned}$$

For small values of l we can approximate

$$u_l(r) \approx \frac{1}{2}u(r+\frac{1}{4}l) - \frac{1}{2}u(r+\frac{3}{4}l),$$

or, taking into account the statistical homogeneity of the flow, we get

$$u_l(r) \approx \frac{1}{2}u(r) - \frac{1}{2}u(r+\frac{l}{2}). \tag{53}$$

So, the power-law behavior (52), viz

$$u_l(r) = -\frac{1}{2}\delta u(r, \frac{l}{2}) \sim l^h,$$

is just a particular case of a local regularity of wavelet coefficients, with the Haar function (51) being used as a basic wavelet. As it was shown in general settings [36], the wavelet coefficients $W_\psi(a,x)[f]$ of a square-integrable function $f(x)$, which has the Lipshitz-Hölder exponent h at the point $x = x_0$, behave as $|W_\psi(a,x)[f]| \sim a^h$ inside the cone $|x - x_0| < const$ for any admissible wavelet ψ which satisfies the regularity condition

$$\int_{-\infty}^{\infty} dx(1+|x|)|\psi(x)| < \infty. \tag{54}$$

The condition (54) is rather loose, and in physical settings one can always assume that it holds for any analyzing function used to measure the pulsations of scale l. So, the second Kolmogorov hypothesis can be formulated as follows:

H2: Generalized second Kolmogorov hypothesis of similarity Under the same assumption as for H1 the turbulent flow is self-similar in small (but still $l \gg \nu^{\frac{3}{4}}\epsilon^{-\frac{1}{4}}$) scales in the sense, that the pulsations of the turbulent velocity defined as

$$u_l(b) = \int \frac{1}{l}\bar{\psi}\left(\frac{x-b}{l}\right)u(x)dx,$$

where $\psi(x)$ is any analyzing function satisfying the admissibility condition (8) and the regularity condition (54), have the following power-law behavior

$$|u_l(b)|^2 \overset{law}{=} l^{2h}, \quad h = \frac{1}{3}, \tag{55}$$

for all spatial points b occupied by turbulent media.

6 Concluding remarks

In this paper we have a stochastic hydrodynamics approach to the NSE based on a wavelet decomposition of both the velocity field and the stirring force used to balance the energy dissipation. In spite of a good deal of papers devoted do different choices of the stirring force for the NSE in wavenumber space – see *e.g.* . [15, 37, 17] and references therein – what is essentially new in our approach is the definition of random force by the correlation function of its wavelet coefficients. Establishing the force correlator in the space of wavelet coefficients, we have got it easy to get rid of loop divergences in the stochastic perturbation expansion, having at the same time desired physical properties of the forcing (energy injection at a given scale). Therefore, a new UV-finite framework is constructed for statistical hydrodynamics. This is a technical framework for analytical evaluation of the statistical characteristics of turbulent fluctuations, such as their correlation and response functions, by means of continuous wavelet transform. Deriving physical consequences, such as energy cascade between scales or scale-dependent corrections to response function, we specially did not touch the renormalization group aspects [15, 38] of the problem and multifractal formalism [39, 40, 19]. In fact, both are related. The former is a generalization of the description of hydrodynamic turbulence in terms of differential equations to the description in measure settings. Partially, the relation of wavelets and RG in turbulence description is discussed in [41], and will be studied in more details in connection to multifractal properties of hydrodynamic turbulence. Besides, the possible comparison with the RG based classification of asymptotic regimes of isotropic turbulence [17, 24] can be considered as another perspective of the proposed method.

The study of hydrodynamic turbulence by methods of stochastic differential equations and those of quantum field theory has at least half a century history. Regardless phenomenologically clear and widely accepted Kolmogorov [1] theory of fully developed turbulence, still there are discussions on the preference of either differential equations, or field theoretic methods based on renormalization group, or multifractal approach to describe the turbulence in an incompressible fluid.

As it concerns the physical interpretation of the velocity field wavelet coefficients, by this paper we intended to say that stochastic nature of spatially extended hydrodynamic turbulence prescribes a certain kind of "wave-particle dualism" to the turbulent phenomena, in a sense, that the answer we get depends on the basic functions used to describe the turbulence. If the basis of plain waves was chosen, there are no fair reasons to comply about $k \to 0$ behavior or paradoxes: what we get is what we set. Alternatively, if we want to have an analytical description of the spatially extended turbulence compatible with the Kolmogorov phenomenology of local turbulent pulsations of given scales, we need to set a functional basis that respects the scale locally. This is the wavelet decomposition.

The Fourier transform, being essentially nonlocal, apparently does not fit the above mentioned requirements, but the windowed Fourier transform, or wave packet decomposition used by V.Zimin [4] and T.Nakano for the analysis of turbulence does, and possibly there is only a technical difference between our approach and that of Nakano [5]. However, it is important to emphasize that the incorporation of the basic function ψ into consideration makes us to admit that the definition of the local fluctuations of a given scale is not completely objective and depends on means of observation. As it was shown, the Kolmogorov hypotheses (K41) were easily rewritten in the wavelet framework for a multiscale description of turbulence.

Acknowledgement

The author is thankful to Drs. M.Hnatich, J.Honkonen, M.Jurcisin and O.Chkhetiani for useful discussions. This work is supported in part by Russian Foundation for Basic Research, project 03-01-00657.

A Calculation of one-loop diagrams

A.1 Response function

To calculate the one-loop diagram in the response function, we introduce the following tensor structure

$$L(k,q,a,s) = -m(k,a,b,c)o(q,b,l)m(p,c,l,s), \tag{56}$$

where the summation over all dummy indices is assumed. The following notation for the orthogonal projector and the vertex (4) is used ($p = k - q$):

$$o(q,b,l) = \delta_{bl} - \frac{q_b q_l}{q^2}, \quad m(p,c,l,s) = \frac{1}{2}\big[p_l o(p,c,s) + p_s o(p,c,l)\big].$$

After all convolutions in matching pairs of indices, substituting $\mathbf{k}\cdot\mathbf{q} = kq\mu$, where $\mu \equiv \cos\theta$ is the cosine of the angle between k and q, we get

$$\begin{aligned} L(k,q,a,s) &= \delta_{as}\frac{k^2(\mu^2-1)}{4} + k_a k_s \frac{2k^2\mu^2 + 2\mu kq(1-2\mu^2) + p^2(1-4\mu^2)}{4p^2} \\ &+ k_a q_s \frac{-k^2\mu^2 + kq\mu(2\mu^2-1) + \mu^2 p^2}{2p^2} + q_a k_s \frac{-2k^3\mu + 2k^2 q(2\mu^2-1) + 3k\mu p^2}{4p^2 q} \\ &+ q_a q_s \frac{k^3\mu + k^2 q(1-2\mu^2) - k\mu p^2}{2qp^2}. \end{aligned}$$

After substitution $p^2 = k^2 - 2kq\mu + q^2$ in the numerators and some algebraic simplification

$$L(k,q,a,s) = \frac{1}{2}\left[T_1 k^2 \delta_{as} + T_2 k_a k_s + T_3 k_a q_s + T_4 q_a k_s + T_5 q_a q_s\right]$$

where

$$\begin{aligned}
T_1 &= \frac{\cos^2\theta - 1}{2}, \\
T_2 &= \frac{k^2 + q^2 - 2\cos^2\theta(k^2 + 2q^2) + 4kq\cos^3\theta}{2p^2}, \\
T_3 &= \frac{q^2\cos^2\theta - kq\cos\theta}{p^2}, \\
T_4 &= \frac{k^3\cos\theta - 2k^2 q\cos^2\theta + 3kq^2\cos\theta - 2k^2 q}{2qp^2}, \\
T_5 &= \frac{k^2 - kq\cos\theta}{p^2}.
\end{aligned}$$

To calculate the whole one-loop integral contribution to the response function, the tensor structure $L(k,q,a,s)$ is multiplied by the integral over the frequency component

$$\int_{-\infty}^{\infty} \frac{dq_0}{2\pi} |G_0(q)|^2 G_0(k-q) = \frac{1}{2\nu^2 \mathbf{q}^2} \frac{1}{\frac{k_0}{\imath\nu} + \mathbf{q}^2 + (\mathbf{k}-\mathbf{q})^2}. \tag{57}$$

Using this structure we easily get the 1PI one-loop contribution to the response function

$$\begin{aligned}
\hat{u}_i(k) &= G_0(k)\hat{\eta}_i(k) + G_0^2(k) 4\lambda^2 \int q^{d-1} dq d\theta \sin^{d-2}\theta d\phi \frac{1}{2\nu^2\mathbf{q}^2} \frac{\Delta(\mathbf{q})}{\frac{k_0}{\imath\nu} + \mathbf{q}^2 + \mathbf{p}^2} \\
&\quad \frac{1}{2}\left[T_1 k^2 \delta_{as} + T_2 k_a k_s + T_3 k_a q_s + T_4 q_a k_s + T_5 q_a q_s\right]\hat{\eta}_s(k).
\end{aligned}$$

In $d = 3$ we always assume $\mathbf{k} = k\mathbf{e}_z$, with θ being the polar angle $\mathbf{k}\cdot\mathbf{q} = kq\cos\theta$ and ϕ being the azimuthal angle. Let us evaluate the integral (58) in $d = 3$

$$\begin{aligned}
\hat{u}_i(k) &= G_0(k)\hat{\eta}_i(k) + \lambda^2 G_0^2(k) S_2 \int q^2 dq d\theta \sin\theta \frac{1}{\nu^2\mathbf{q}^2} \frac{\Delta(\mathbf{q})}{\frac{k_0}{\imath\nu} + \mathbf{q}^2 + \mathbf{p}^2} \\
&\quad \mathrm{diag}(T_1 k^2 + \frac{q^2}{2}\sin^2\theta T_5, T_1 k^2 + \frac{q^2}{2}\sin^2\theta T_5, 0)\Big]\hat{\eta}_i(k)
\end{aligned}$$

Here diag() means diagonal matrix, where the first two terms give the transversal contribution to viscosity, and the last term, giving the longtitudal contribution, vanishes identically:

$$k^2(T_1 + T_2) + kq\cos\theta(T_3 + T_4) + q^2\cos^2\theta T_5 = 0.$$

Finally, after integration in angle variable $\cos\theta$, we get

$$\hat{u}_i(k) = G_0(k)\hat{\eta}_i(k) + \lambda^2 G_0^2(k)\frac{S_2}{\nu^2}\int_0^\infty dq\Delta(\mathbf{q})\mathrm{diag}(R(k/q), R(k/q), 0)\hat{\eta}_i(k) \tag{58}$$

where

$$\begin{aligned} R(x) &= \frac{1}{32x^3}\Big[12x - 16x^3 - 8x^5 + (x^2-1)^3 \ln\left(\frac{x-1}{x+1}\right)^2 \\ &+ (3x^6 - 2x^4 + 12x^2 - 8)\ln\frac{2+2x+x^2}{2-2x+x^2}\Big]. \end{aligned} \tag{59}$$

A.2 Correlation function

Similarly, for the tensor structure of the one-loop contribution to the response function, we evaluate the tensor structure of the one-loop contribution to the pair correlator

$$C(k, q, a, s) = m(k, a, b, c)m(k, s, l, t)o(q, b, l)o(p, c, t).$$

After algebraic simplification

$$\begin{aligned} C(k, q, a, s) &= \delta_{as}\frac{-k^4 + 2k^3\mu q + k^2(-\mu p^2 - \mu^2 q^2 + 2p^2)}{4p^2} \\ &+ k_a k_s \frac{k^2(1+\mu^2) - 2kq\mu(1+2\mu^2) + 2(2\mu^4 q^2 - p^2)}{4p^2} \\ &+ (k_a q_s + q_a k_s)\frac{-k^3\mu + 4k^2 q\mu^2 + k\mu(-4\mu^2 q^2 + p^2 + q^2)}{4p^2 q} \\ &+ q_a q_s \frac{k^4 - 4k^3\mu q + k^2(4\mu^2 q^2 - p^2 - q^2)}{4p^2 q^2}. \end{aligned} \tag{60}$$

The trace of this tensor structure, *i.e.* $c_2(k, q) = \sum_a C(k, q, a, a)$, required for the energy spectra evaluation, is equal to

$$c_2(k, q) = \frac{(1-\mu^2)k^2}{4p^2}\big[k^2(d-1) - 2kq\mu d + 2q^2(d + 2\mu^2 - 2)\big]. \tag{61}$$

In the important case of $d = 3$ this gives

$$c_2(k, q) = \frac{(1-\mu^2)k^2}{2(k^2 + q^2 - 2kq\mu)}\big[k^2 - 3kq\mu + q^2(1 + 2\mu^2)\big]. \tag{62}$$

The integral in frequency argument of the product of squared response functions in the integral (34) gives in the limit of zero frequency $k_0 \to 0$:

$$\int_{-\infty}^{\infty} \frac{dq_0}{2\pi} |G_0(q)|^2 |G_0(k-q)|^2 \to \frac{1}{2\nu^3} \frac{1}{\mathbf{q}^2(\mathbf{k}-\mathbf{q})^2} \frac{1}{\mathbf{q}^2+(\mathbf{k}-\mathbf{q})^2}. \tag{63}$$

In the important case of $d=3$, with the Mexican hat wavelet ($n=2$) taken for definiteness, the integral over the angle variable $\mu = \cos\theta$ can be evaluated analytically for the single band random force (25). The angle integration in (38) gives

$$C_{eff}^{d=3,n=2}(k) = \lambda^2 \frac{k^2 |G_0(k)|^2 4\pi a_0^8 D_0^2}{2\nu^3} \int_0^\infty dq q^4 i_c(k/q), \tag{64}$$

where

$$\begin{aligned} i_c(x) &= \int_{-1}^{1} dy e^{-a_0^2 q^2 (2+x^2-2xy)} \frac{(1-y^2)(1+x^2-3xy+2y^2)}{2+x^2-2xy} = F(x) + F(-x) \quad (65) \\ F(x) &= \frac{1}{8x^5}\Big[\frac{e^{-a_0^2 q^2(2+x^2-2x)}}{a_0^8 q^8}\big(-3-2a_0^2q^2(-1-3x+x^2)+2a_0^4q^4(-1-2x-2x^2+2x^3) \\ &+ 2a_0^6 q^6 (2+2x+x^2)\big) + 2(4+x^4)\mathrm{ExpIntEi}\left(-a_0^2q^2(2+x^2-2x)\right)\Big]. \quad (66) \end{aligned}$$

In the limit of small wave numbers $x = \frac{k}{q} \to 0$ this gives

$$\int_0^\infty dq q^4 \lim_{x\to 0} i_c(x) = \frac{7}{80} \frac{\sqrt{\frac{\pi}{2}}}{a^5}.$$

Therefore the effective correlator tends to

$$C_{eff}^{d=3,n=2}(k\to 0) = \lambda^2 \frac{7}{40} \frac{k^2 |G_0(k)|^2 \pi^{\frac{3}{2}} a_0^3 D_0^2}{\nu^3 \sqrt{2}}.$$

References

[1] A.N. Kolmogorov. The local structure of the turbulence in incompressible viscous fluid for very large Reynolds numbers. *Dokl. Akad. Nauk SSSR*, 30:9–13, 1941. Reprinted in Proc. R. Soc. Lond. A434, 9-13,1991.

[2] A.N. Kolmogorov. A refinment of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at a high Reynolds number. *J.Fluid.Mech*, 13:82–85, 1962.

[3] A.M. Obukhov. Some specific features of atmospheric turbulence. *J.Fluid.Mech*, 13:77–81, 1962.

[4] V.D. Zimin. Hierarchic turbulence model. *Izv. Atmos. Ocean. Phys.*, 17:941–946, 1981.

[5] T. Nakano. Direct interaction approximation of turbulence in wave packet representation. *Phys. Fluids*, 31(6):1420–1430, 1988.

[6] C. Meneveau. Analysis of turbulence in the orthonormal wavelet representation. *J.Fluid.Mech*, 232:469–520, 1991.

[7] M. Vergassola and U. Frisch. Wavelet transforms of self-similar processes. *Physica D*, 54:58–64, 1991.

[8] M. Farge. Wavelets and their application to turbulence. *Annual review of fluid mechanics*, 24:395–457, 1992.

[9] N.M. Astaf'eva. Wavelet analysis: basic theory and some applications. *Uspekhi fizicheskih nauk*, 166(11):1146–1170, 1994. (in Russian).

[10] J.M. Restero and G.K. Leaf. Wavelet-Galerkin discretization of hyperbolic equations. *J.Comp.Phys.*, 122:118–128, 1995.

[11] A. Grossman and M Löhden. Wavelet analysis of the Navier-Stokes flow. *Z.Phys.B*, 100:137–147, 1996.

[12] K. Shneider and M. Farge. *Wavelet approach for modelling and computing turbulence*, volume 1998-05 of *Advances in turbulence modelling.* Von Karman Institute for Fluid Dynamics, Bruxelles, 1998.

[13] J. Lewalle. Energy dissipation in the wavelet-transformed Navier-Stokes equations. *Phys. Fluids A*, 5:1512–1513, 1993.

[14] J. Lewalle. Wavelet transforms of the Navier-Stokes equations and the generalized dimensions of turbulence. *App. Sci. Res.*, 51:109–113, 1993.

[15] D. Forster, D.R. Nelson, and M.J. Stephen. Long-distance and long-time properties of a randomly stirred fluid. *Phys. Rev. A*, 16(2):732–749, 1977.

[16] V. Yakhot and S. Orszag. Renormalization group analysis of turbulence. *Phys. Rev. Lett.*, 57(14):1722–1724, 1986.

[17] L.Ts. Adzhemyan, N.V. Antonov, and A.N. Vasiliev. *Field Theoretic Renormalization Group in Fully Developed Turbulence.* Gordon and Breach, 1999.

[18] K. Gawedzki, A. Kupianinen, and M. Vergassola. A program on developed turbulence. Erwin Schrödinger International Institute for Mathematical Physics, 2002. http://www.esi.ac.at/Programs/turbulence2002.html.

[19] J.F. Muzy, E. Bacry, and A. Arneodo. Wavelets and multifractal formalism for singular signals: Application to turbulence data. *Phys. Rev. Lett.*, 67:3515–3518, 1991.

[20] A. Arneodo, F. Argoul, E. Bacry, J. Elezgaray, and J.-F. Muzy. *Ondelettes, multifractales et turbulence.* Diderot, Paris, 1995.

[21] S.S. Moiseyev, A.V. Tur, and V.V. Yanovsky. The spectra and means of excitation of turbulence in an compressible fluid. *JETP*, 71(3):1062–1073, 1976.

[22] M. Chertkov, G. Falkovich, I. Kolokolov, and Lebedev. V. Normal and anomalous scaling of the fourth-order correlation function of a randomly advected passive scalar. *Phys. Rev. E*, 52(5):4924–4941, 1995.

[23] G. Gawedzki and A. Kupiainen. Anomalous scaling of a passive scalar. *Phys. Rev. Lett.*, 75(21):3834–3837, 1995.

[24] N.V. Antonov, M. Hnatich, J. Honkonen, and M. Jurcisin. Turbulence with pressure:Anomalous scaling of a passive vector field. *Phys. Rev. E*, 68:046306–1–25, 2003.

[25] S.A. Orszag and G.S. Patterson. Numerical simulation of turbulence. In M. Rosenblatt and C.V. Van Atta, editors, *Statistical models and turbulence*, volume 12 of *Lecture notes in physics*, pages 127–147. Springer, 1972.

[26] J.C. Bowman, B.A. Shadwick, and P.J. Morrison. Spectral reduction: A statistical description of turbulence. *Phys. Rev. Lett.*, 83(26):5491–5494, 1999.

[27] U. Frish. *Turbulence*. Cambridge University Press, 1995.

[28] D.A. Kennedy and S. Corrsin. *J.Fluid. Mech.*, 10:366, 1961.

[29] M. Germano. Turbulence: the filtering approach. *J. Fluid. Mech.*, 238:325–336, 1992.

[30] J. Lewalle. Wavelet transforms of some equations of fluid mechanics. *Acta Mechanica*, 104:1–25, 1994.

[31] W.D. McComb. *The Physics of turbulence*. Clarendon Press, Oxford, 1990.

[32] M.V. Altaisky. Langevin equation with scale-dependent noise. *Doklady Physics*, 48(9):478–480, 2003.

[33] Wyld H. W. Formulation of the theory of turbulence in an incompressible fluid. *Annals of Physics*, 14(2):143–165, 1961.

[34] A. Latto, H.L. Resnikoff, and E. Tenenbaum. The evaluation of connection coefficients of compactly supported wavelets. In *French-USA Workshop on Wavelets and Turbulence*, New York, 1991. Springer-Verlag.

[35] S. Qian and J. Weiss. Wavelets and the numerical solution of partial differential equations. *Journal of Computational Physics*, 106:155–175, 1993.

[36] M. Holschneider and P. Tchamitchan. Regularite locale de la fonction "non-differentiable" de Riemann. In Lemarie P.G., editor, *Les Ondeletes*, pages 102–124. Springer, Berlin, 1990.

[37] V. Eswaran and S.B. Pope. Direct numerical simulations of the turbulence mixing of a turbulence mixing a passive scalar. *Phys. Fluids*, 31:506–520, 1988.

[38] J.D. Fournier and U. Frisch. Remarks on the renormalization group in statistical fluid dynamics. *Phys. Rev. A*, 28:1000–1002, 1983.

[39] R. Benzi, G. Paladin, G. Parizi, and A. Vulpiani. On multifractal nature of fully developed turbulence and chaotic systems. *J. Phys.*, 17:3521–3531, 1984.

[40] A. Arneodo, G. Grasseau, and M. Holshneider. Wavelet transform of multifractals. *Phys. Rev. Lett.*, 61:2281–2284, 1988.

[41] M. Altaisky. Renormalization group and geometry. In B.G. Sidharth and M.V. Altaisky, editors, *Frontiers of Fundamental Physics 5*, Hyderabad, India, 2003.